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2006 J. Phys. A: Math. Gen. 39 11657

(http://iopscience.iop.org/0305-4470/39/37/020)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **39** (2006) 11657–11681 doi:10.1088/0305-4470/39/37/020

Manifestly gauge-invariant QCD

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Received 29 June 2006
Published 29 August 2006
Online at stacks.iop.org/JPhysA/39/11657

Abstract

Building on recent work in $SU(N)$ Yang–Mills theory, we construct a manifestly gauge-invariant exact renormalization group for QCD. A gauge-invariant cut-off is constructed by embedding the physical gauge theory in a spontaneously broken $SU(N|N)$ gauge theory, regularized by covariant higher derivatives. Intriguingly, the construction is most efficient if the number of flavours is a multiple of the number of colours. The formalism is illustrated with a very compact calculation of the one-loop β-function, achieving a manifestly universal result and without fixing the gauge.

PACS numbers: 11.15.Tk, 11.10.Hi, 12.38.−t

1. Introduction

One of the hallmarks of QCD is the qualitatively very different behaviours observed in the high and low energy domains. In the former case, the theory exhibits asymptotic freedom, allowing phenomena to be described in terms of weakly interacting quarks and gluons. Since the coupling is small, calculations can be performed in perturbation theory. However, as the energy scale is lowered, so the coupling strength increases, ultimately causing the quarks and gluons to be bound together into hadrons. The failure of perturbative techniques to capture this behaviour presents a stern challenge.

A promising approach to extracting information from the strongly coupled (or non-perturbative) domain of quantum field theories is the exact renormalization group (ERG) [1–3], the continuous version of Wilson's RG. The central feature of the ERG is the implementation of a momentum cut-off, Λ, in the theory in such a way that the physics at this scale—which is encoded in the Wilsonian effective action, S_Λ—is described in terms of parameters relevant to this scale. The ERG equation determines how S_Λ evolves with Λ, thereby linking physics at different energy scales. Consequently, an ERG for QCD has the potential to provide access to the strongly coupled regime.

In addition to providing a powerful framework for addressing a wealth of non-perturbative problems in a range of settings (see [4–12] for reviews), a particular advantage conferred by the ERG is the huge freedom inherent in its construction [13]. In the context of gauge theories, this freedom can be exploited to construct manifestly gauge-invariant ERGs [14–19] (for a comprehensive review of the alternatives, see [20]). Whilst being of obvious novelty value, manifest gauge invariance also provides both technical and conceptual benefits. From the technical standpoint, the gauge field is protected from field strength renormalization and the Ward identities take a particularly simple form since the Wilsonian effective action is built only from gauge-invariant combinations of the covariant derivative, even at the quantum level [15]. In addition, the difficult technical issue of Gribov copies [21] is entirely avoided. Conceptually, a strong case can be made for manifest gauge invariance being the natural language to describe non-perturbative phenomena, not least because all conclusions drawn will be completely gauge independent.

The majority of work into this scheme has, so far, focused on constructing [14–19], testing [16–18, 22] and refining [23–25] the formalism. Most recently, however, real progress has been made in understanding how to compute objects of particular interest, specifically, the expectation values of gauge-invariant operators [26]. Moreover, crucial steps in this procedure have a non-perturbative extension [27]. Given these developments we feel that it is timely to extend the framework to incorporate quarks, in anticipation of application to non-perturbative QCD.

The strategy we adopt is to build directly on to the $SU(N)$ Yang–Mills construction, which we briefly describe. Recall that the implementation of a gauge-invariant cut-off comprises two ingredients [28]. First, we apply covariant higher derivative regularization. However, as is well known [29], this is insufficient to completely regularize the theory, since certain one-loop divergences slip through. The solution we employ is to instead apply the covariant higher derivative regularization to a spontaneously broken $SU(N|N)$ gauge theory, into which the physical $SU(N)$ gauge theory has been embedded. The heavy fields arising from the symmetry breaking act as Pauli–Villars (PV) fields, supplementing the covariant higher derivatives to furnish a complete regularization of the physical theory.

The symmetry breaking is carried by a Higgs field, $\mathcal{C}$. Upon acquiring a vacuum expectation value (vev), this field breaks $SU(N|N)$ down to its bosonic subgroup, $SU(N) \times SU(N) \times U(1)$. One of these $SU(N)$ symmetries is identified with the symmetry of the physical gauge field, A^1; the other is identified with an unphysical field, A^2, which we note comes with wrong sign action. (We can effectively ignore $U(1)$, as we will see later.) Besides A^1, only A^2 remains massless upon spontaneous symmetry breaking, all other fields picking up a mass of order Λ. As we send Λ to infinity, all effective interactions between A^1 and A^2 vanish and so the non-unitary A^2 sector decouples and can be ignored [28].

The most obvious way to add quarks is to embed them in the fundamental representation of $SU(N|N)$. However, as we will see, the structure of the $SU(N|N)$ group in fact forbids this program. Instead, we embed N quarks into a field which transforms as a bifundamental under $U(N|N)$. This essentially corresponds to a gauging of not only the physical colour symmetry but also, in an entirely unphysical way, a flavour symmetry. The unphysical fields which accompany the physical quarks are given a mass of order the cut-off and so act as precisely the set of PV fields we require for the theory to be regularized. In this way, we are able to incorporate quarks in multiples of N, such that the elements of each set have the same mass. To obtain quarks with arbitrary masses, we now modify the Higgs sector to break the unphysical $SU(N)$ symmetry completely.

Quite apart from the ERG, the inclusion of quarks in the $SU(N|N)$ regularizing scheme is of interest in itself, since it allows the construction of a real, gauge-invariant cut-off in QCD. In the case of pure $\mathcal{N} = 4$ super Yang–Mills, a dual picture of this was recently constructed [30], providing a concrete understanding of how the radial direction in the AdS/CFT correspondence plays the role of a gauge-invariant measure of energy scale. It will be interesting to see whether the inclusion of quarks in the $SU(N|N)$ regularizing scheme leads to a new way to introduce quarks in the dual picture.

Having incorporated massive quarks into the regularization framework, it is now a straightforward matter to generalize our manifestly gauge-invariant flow equation from $SU(N)$ Yang–Mills to QCD. To understand how we go about doing this, we review the structure of general flow equations. One of the key ingredients of any flow equation is that the partition function (and hence the physics derived from it) is invariant under the flow. As a consequence of this, the family of flow equations for some generic fields, φ, follows from [13–15, 19]

$$-\Lambda\partial_\Lambda \,\mathrm{e}^{-S[\varphi]} = \int_x \frac{\delta}{\delta\varphi(x)}(\Psi_x[\varphi]\,\mathrm{e}^{-S[\varphi]}), \tag{1.1}$$

where the functional Ψ parameterizes how the high energy modes are averaged over (and we have written S_Λ as just S). The total derivative on the right-hand side ensures that the partition function $Z = \int \mathcal{D}\varphi\ \mathrm{e}^{-S}$ is invariant under the flow.

Taking φ to represent a single scalar field, we can use [32]

$$\Psi_x = \frac{1}{2}\int_y \dot{\Delta}_{xy}\frac{\delta\Sigma_1}{\delta\varphi(y)}, \tag{1.2}$$

where $\dot{\Delta}$ is an ERG kernel and $\Sigma_1 = S - 2\hat{S}$, with $\hat{S}$ being the seed action [17–19, 22, 31–33]: a functional which respects the same symmetries as the Wilsonian effective action, S, and has the same structure. Physically, the seed action can be thought of as (partially) parameterizing a general Kadanoff blocking in the continuum [13, 19] and acts as an input to our flow equation.

There is a deep relationship between the kernel and the classical, two-point vertices. Specifically, if we set the seed action classical, two-point vertex equal to its Wilsonian effective action counterpart, $S_0^{\varphi\varphi}$, then we find that [32]

$$S_0^{\varphi\varphi}\Delta = 1, \tag{1.3}$$

where Δ is the integrated kernel a.k.a. effective propagator:

$$\dot{\Delta} \equiv -\Lambda\partial_\Lambda\Delta.$$

Equation (1.3) is the effective propagator relation [17] and is at the heart of the computational technique employed within our approach [17–19, 22–26, 32, 33].

Now suppose that we consider a flow equation for some set of fields, rather than a single field. It is highly desirable to insist on an effective propagator relation for each individual field which means that, in general, the number of effective propagators—and hence the number of kernels—must equal the number fields. This observation holds the key to generalizing our flow equation for $SU(N)$ Yang–Mills to one appropriate for QCD.

To construct an ERG for $SU(N)$ Yang–Mills, we use the template (1.1), covariantize the relationship (1.2) and incorporate the $SU(N|N)$ regularizing structure [17, 19]. By defining the covariantization appropriately (we will review this in section 4.1), we can ensure that an effective propagator exists for each of the broken phase fields. Note, however, that the form of the effective propagator relation is different to (1.3) in the gauge sector, as a consequence of the manifest gauge invariance:

$$S_{0\mu\ \alpha}^{\ A^1A^1}(p)\Delta_{\alpha\ \nu}^{A^1A^1}(p) = \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}. \tag{1.4}$$

$S_{0\mu\ \alpha}^{\ A^1A^1}(p)$ is the two-point classical vertex in the A^1 sector, carrying momentum p, and $\Delta_{\alpha\ \nu}^{A^1A^1}(p)$ is the associated effective propagator. Thus, we see that the effective propagator is the inverse of the classical, two-point vertex only in the transverse space; equivalently, it is the inverse only up to a remainder which we call a 'gauge remainder'.

To add quarks, we again use the template (1.1), covariantize the relationship (1.2) and incorporate the $SU(N|N)$ regularizing structure, but this time the covariantization is designed appropriately for the quarks. To allow independent quark masses, we update the Higgs sector of the flow equation and modify the covariantizations in the gauge and quark sectors to ensure that there are enough independent kernels to cope with the breaking of the unphysical $SU(N)$ symmetry.

Anticipating that the flow equation for QCD is most efficiently stated via its diagrammatic representation [17–19, 22–25], we could jump straight to this diagrammatic form, using it to hide non-universal details such as the complicated form of the covariantization. However, before doing this, we will provide an explicit example of a valid covariantization, for completeness. Nevertheless, it should be understood that this is just one of an infinite number of choices which we can make but which, in practice, we never do: in actual calculations, we implicitly work with an infinite number of flow equations. The reason we can do this is that there exists a powerful diagrammatic calculus which enables us to perform general computations, almost entirely at the diagrammatic level [17, 25, 26], in a way such that non-universal details cancel out. Indeed, this calculus has been employed in pure $SU(N)$ Yang–Mills to give a diagrammatic expression for the β-function, from which the universal answer (at least at one and two loops) can be directly extracted. This formula can be trivially adapted to QCD and we will use this to perform a very compact computation of the one-loop β-function.

The outline of this paper is as follows. In section 2, we review the regularization of $SU(N)$ Yang–Mills via $SU(N|N)$ Yang–Mills. In section 3, we add the quarks, first seeing why we cannot embed them in the fundamental of $SU(N|N)$ and then describing how we can instead embed them using a more elaborate scheme. We conclude this section by showing how to give the quarks independent masses. In section 4, we review the construction of a manifestly gauge-invariant flow equation for $SU(N)$ Yang–Mills and then adapt it for QCD. In section 5, we give a diagrammatic expression for the one-loop β-function which reproduces the universal result in the case that the quarks are massless. Finally, in section 6, we conclude.

2. Regularizing $SU(N)$ Yang–Mills

2.1. Embedding in $SU(N|N)$ Yang–Mills

Throughout this paper, we work in Euclidean dimension, D. We regularize $SU(N)$ Yang–Mills by embedding it in spontaneously broken $SU(N|N)$ Yang–Mills, which is itself regularized by covariant higher derivatives [28]. The supergauge field, $\mathcal{A}_\mu$, is valued in the Lie superalgebra and, using the defining representation, can be written as a Hermitian supertraceless supermatrix (the supertrace of a supermatrix is defined as the trace of the top block diagonal element minus the trace of the bottom block diagonal element):

$$\mathcal{A}_\mu = \begin{pmatrix} A^1_\mu & B_\mu \\ \bar{B}_\mu & A^2_\mu \end{pmatrix} + \mathcal{A}^0_\mu \mathbb{1}. \tag{2.1}$$

Here, $A^1_\mu(x) \equiv A^1_{a\mu}\tau^a_1$ is the physical $SU(N)$ gauge field, τ^a_1 being the $SU(N)$ generators orthonormalized to $\mathrm{tr}\left(\tau^a_1\tau^b_1\right) = \delta^{ab}/2$, while $A^2_\mu(x) \equiv A^2_{a\mu}\tau^a_2$ is a second, unphysical $SU(N)$ gauge field. The B fields are fermionic gauge fields which will gain a mass of order Λ from the spontaneous symmetry breaking; they play the role of gauge-invariant PV fields, furnishing the necessary extra regularization to supplement the covariant higher derivatives. In order to unambiguously define contributions which are finite only by virtue of the PV regularization, a preregulator must be used in $D = 4$ [28]. We will use dimensional regularization, emphasizing that this makes sense non-perturbatively, since it is not being used to renormalize the theory, but rather as a prescription for discarding surface terms in loop integrals [14, 28].

$\mathcal{A}^0$ is the gauge field for the centre of the $SU(N|N)$ Lie superalgebra. Equivalently, one can write

$$\mathcal{A}_\mu = \mathcal{A}^0_\mu \mathbb{1} + \mathcal{A}^A_\mu T_A, \tag{2.2}$$

where T_A are a complete set of traceless and supertraceless generators normalized as in [28].

The theory is subject to the local invariance:

$$\delta\mathcal{A}_\mu = [\nabla_\mu, \Omega(x)] + \lambda_\mu(x)\mathbb{1}. \tag{2.3}$$

The first term, in which $\nabla_\mu = \partial_\mu - \mathrm{i}\mathcal{A}_\mu$, generates supergauge transformations. Note that the coupling, g, has been scaled out of this definition. It is worth doing this: since we do not gauge fix, the exact preservation of (2.3) means that none of the fields suffer wavefunction renormalization, even in the broken phase [17]. The second term in (2.3) divides out the centre of the algebra. The reason for doing this is as follows. The superfield strength is $\mathcal{F}_{\mu\nu} = \mathrm{i}[\nabla_\mu, \nabla_\nu]$, out of which we construct the kinetic term $\sim \mathrm{str}\, \mathcal{F}^2_{\mu\nu}$. On account of $\mathrm{str}\, \mathbb{1} = 0$ and the fact that $\mathbb{1}$ commutes with everything, it is apparent that $\mathcal{A}^0$ has neither a kinetic term nor any interactions. Consequently, if $\mathcal{A}^0$ were to appear anywhere else in the action, it would act as a Lagrange multiplier and so we forbid its presence. The resulting 'no-$\mathcal{A}^0$ shift symmetry' ensures that nothing depends on $\mathcal{A}^0$ and that $\mathcal{A}^0$ has no degrees of freedom[1]. This will prove important when we come to add quarks.

The spontaneous breaking is carried by a superscalar field

$$\mathcal{C} = \begin{pmatrix} C^1 & D \\ \bar{D} & C^2 \end{pmatrix}.$$

This field is Hermitian but, unlike $\mathcal{A}_\mu$, is not supertraceless and so it is valued in the $U(N|N)$ Lie algebra. Nevertheless, the whole of $\mathcal{C}$ transforms homogeneously under local $SU(N|N)$:

$$\delta\mathcal{C} = -\mathrm{i}[\mathcal{C}, \Omega]. \tag{2.4}$$

It can be shown that, at the classical level, the spontaneous breaking scale (effectively the mass of B) tracks the covariant higher derivative effective cut-off scale, Λ, if $\mathcal{C}$ is made dimensionless (by using powers of Λ) and $\hat{S}$ has the minimum of its effective potential at

$$\langle\mathcal{C}\rangle = \sigma \equiv \begin{pmatrix} \mathbb{1}_N & 0 \\ 0 & -\mathbb{1}_N \end{pmatrix}, \tag{2.5}$$

where $\mathbb{1}_N$ is the $N \times N$ identity matrix.

In this case, the classical action S_0 also has a minimum at (2.5). At the quantum level this can be imposed as a constraint on S by taking $\langle\mathcal{C}\rangle = \sigma$ as a renormalization condition. This ensures that the Wilsonian effective action does not possess any one-point vertices, which can be translated into a constraint on $\hat{S}$ [17, 18]. In the broken phase, D is a super-Goldstone mode (eaten by B in the unitary gauge) whilst C^i are Higgs bosons and can be given a running mass of order Λ [14, 17, 28]. Working in our manifestly gauge-invariant formalism, B and D gauge transform into each other, as we will see in section 2.2.

[1] It is tempting to try and remove $\mathcal{A}^0$ from the algebra. However, we cannot do this directly since although $SU(N|N)$ is reducible it is indecomposable: $\mathcal{A}^0$ is generated by (fermionic) gauge transformations. It is possible to instead modify the Lie bracket [34] but this appears only to complicate matters.

In addition to the coupling, g, of the physical gauge field, the field A^2_μ carries its own coupling, g_2 (in the broken phase), which renormalizes separately [17–19, 22]. It is often useful not to work with g_2 directly but rather with

$$\alpha \equiv g_2^2/g^2. \tag{2.6}$$

The couplings g and α are defined through their renormalization conditions:

$$S[\mathcal{A} = A^1, \mathcal{C} = \sigma] = \frac{1}{2g^2}\,\mathrm{str}\int \mathrm{d}^D x \left(F^1_{\mu\nu}\right)^2 + \cdots, \tag{2.7}$$

$$S[\mathcal{A} = A^2, \mathcal{C} = \sigma] = \frac{1}{2\alpha g^2}\,\mathrm{str}\int \mathrm{d}^D x \left(F^2_{\mu\nu}\right)^2 + \cdots, \tag{2.8}$$

where the ellipses stand for higher dimension operators and the ignored vacuum energy. The field strength tensors in the A^1 and A^2 sectors, $F^1_{\mu\nu}$ and $F^2_{\mu\nu}$, should really be embedded in the top left/bottom right entries of a supermatrix, in order for the supertraces in (2.7) and (2.8) to make sense. We will frequently employ this minor abuse of notation, for convenience.

2.2. Ward identities

The supergauge invariant Wilsonian effective action has an expansion in terms of supertraces and products of supertraces [17]:

$$\begin{aligned} S = \sum_{n=1}^{\infty} \frac{1}{s_n} \int &\mathrm{d}^D x_1 \cdots \mathrm{d}^D x_n \, S^{X_1\cdots X_n}_{a_1\cdots a_n}(x_1,\ldots,x_n)\,\mathrm{str}\, X_1^{a_1}(x_1)\cdots X_n^{a_n}(x_n) \\ &+ \frac{1}{2!}\sum_{m,n=0}^{\infty} \frac{1}{s_n s_m}\int \mathrm{d}^D x_1\cdots \mathrm{d}^D x_n\,\mathrm{d}^D y_1\cdots \mathrm{d}^D y_m \\ &\times S^{X_1\cdots X_n, Y_1\cdots Y_m}_{a_1\cdots a_n, b_1\cdots b_m}(x_1,\ldots,x_n;y_1,\ldots,y_m) \\ &\times \mathrm{str}\, X_1^{a_1}(x_1)\cdots X_n^{a_n}(x_n)\,\mathrm{str}\, Y_1^{b_1}(y_1)\cdots Y_m^{b_m}(y_m) + \cdots \end{aligned} \tag{2.9}$$

where $X_i^{a_i}$ and $Y_j^{b_j}$ are any of the broken phase fields, with a_i and b_j being Lorentz indices or null, as appropriate. The vacuum energy is ignored. We take only one cyclic ordering for the lists $X_1\cdots X_n$, $Y_1\cdots Y_m$ in the sums over n, m. If any term is invariant under some nontrivial cyclic permutations of its arguments, then s_n (s_m) is the order of the cyclic subgroup, otherwise $s_n = 1$ ($s_m = 1$).

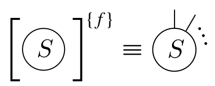
The momentum space vertices are written as

$$S^{X_1\cdots X_n}_{a_1\cdots a_n}(p_1,\ldots,p_n)\,(2\pi)^D\,\delta\left(\sum_{i=1}^{n} p_i\right) = \int \mathrm{d}^D x_1\cdots \mathrm{d}^D x_n\,\mathrm{e}^{-\mathrm{i}\sum_i x_i\cdot p_i}\,S^{X_1\cdots X_n}_{a_1\cdots a_n}(x_1,\ldots,x_n),$$

where all momenta are taken to point into the vertex. We employ the shorthand

$$S^{X_1X_2}_{a_1a_2}(p) \equiv S^{X_1X_2}_{a_1a_2}(p,-p).$$

Since we will ultimately be giving the flow equation for QCD via its diagrammatic representation, it is useful at this stage to introduce diagrammatics for the action (2.9). The *vertex coefficient functions* belonging to the action (2.9) have a simple diagrammatic representation:



(2.10)

represents all vertex coefficient functions corresponding to all cyclically independent orderings of the set of broken phase fields, $\{f\}$, distributed over all possible supertrace structures. For example,

$$\left[\textcircled{S}\right]^{C^1C^1} \tag{2.11}$$

represents both the coefficient functions $S^{C^1C^1}$ and S^{C^1,C^1} which, from (2.9), are associated with the supertrace structures $\operatorname{str} C^1C^1$ and $\operatorname{str} C^1 \operatorname{str} C^1$, respectively. (We have suppressed the momentum arguments.) Similarly,

$$\left[\textcircled{S}\right]^{A^1A^1C^1}_{\mu\nu}$$

represents $S^{A^1A^1C^1}_{\mu\;\nu}$, $S^{A^1A^1C^1}_{\nu\;\mu}$ and $S^{A^1A^1,C^1}_{\mu\;\nu}$. (There are no vertices which correspond to a trace of a single A^1, since $\operatorname{str} A^1 = 0$.)

The (un)broken gauge transformations follow from splitting Ω into its block components

$$\Omega = \begin{pmatrix} \omega^1 & \tau \\ \bar{\tau} & \omega^2 \end{pmatrix} + \Omega^0 1\!\!1$$

and expanding out (2.3) and (2.4) (we are not interested in the no-$\mathcal{A}^0$ symmetry, here). For this purpose, it is useful to combine the fields A^1 and A^2 with the block diagonal components of $\mathcal{A}_0 1\!\!1$. We denote the resultant fields by $\tilde{A}^1$ and $\tilde{A}^2$, though note that sometimes $\mathcal{A}^0$ contributions can cancel out between terms. This gives the unbroken $SU(N)\times SU(N)\times U(1)$ transformations [17]

$$\begin{aligned}
\delta\tilde{A}^1_\mu &= D^1_\mu\cdot\omega^1 + \partial_\mu\Omega^0 1\!\!1_N & \delta\tilde{A}^2_\mu &= D^2_\mu\cdot\omega^2 + \partial_\mu\Omega^0 1\!\!1_N \\
\delta B_\mu &= -\mathrm{i}(B_\mu\omega^2 - \omega^1 B_\mu) & \delta\bar{B}_\mu &= -\mathrm{i}(\bar{B}_\mu\omega^1 - \omega^2\bar{B}_\mu) \\
\delta C^1 &= -\mathrm{i}C^1\cdot\omega^1 & \delta C^2 &= -\mathrm{i}C^2\cdot\omega^2 \\
\delta D &= -\mathrm{i}(D\omega^2 - \omega^1 D) & \delta\bar{D} &= -\mathrm{i}(\bar{D}\omega^1 - \omega^2\bar{D})
\end{aligned} \tag{2.12}$$

and the broken fermionic gauge transformations

$$\begin{aligned}
\delta\tilde{A}^1_\mu &= -\mathrm{i}(B_\mu\bar{\tau} - \tau\bar{B}_\mu) & \delta\tilde{A}^2_\mu &= -\mathrm{i}(\bar{B}_\mu\tau - \bar{\tau}B_\mu) \\
\delta B_\mu &= \partial_\mu\tau - \mathrm{i}\big(A^1_\mu\tau - \tau A^2_\mu\big) & \delta\bar{B}_\mu &= \partial_\mu\bar{\tau} - \mathrm{i}\big(A^2_\mu\bar{\tau} - \bar{\tau}A^1_\mu\big) \\
\delta C^1 &= -\mathrm{i}(D\bar{\tau} - \tau\bar{D}) & \delta C^2 &= -\mathrm{i}(\bar{D}\tau - \bar{\tau}D) \\
\delta D &= -\mathrm{i}(C^1\tau - \tau C^2) - 2\mathrm{i}\tau & \delta\bar{D} &= -\mathrm{i}(C^2\bar{\tau} - \bar{\tau}C^1) + 2\mathrm{i}\bar{\tau},
\end{aligned} \tag{2.13}$$

where $D^{(1,2)} = \partial_\mu - \mathrm{i}A^{1,2}_\mu$ are the covariant derivatives appropriate to the physical gauge field and the unphysical copy and the dot again means action by commutation.

As noted in [17], the manifest preservation of the transformations for $\tilde{A}^1_\mu$ and $\tilde{A}^2_\mu$ in (2.12) protects these fields from field strength renormalization. The remaining fields are similarly protected, as follows from (2.13).

The transformations (2.13) for B_μ and D, $\bar{B}_\mu$ and $\bar{D}$ lead us to define [17][2]

$$F_M = (B_\mu, D), \tag{2.14a}$$

$$\bar{F}_N = (\bar{B}_\nu, -\bar{D}), \tag{2.14b}$$

[2] Actually, these definitions differ from those of [17] by a sign in the fifth component. They are, however, consistent with [18, 19, 22–26].

where M, N are five indices [18, 19]. The summation convention for these indices is that we take each product of components to contribute with unit weight.

Two Ward identities now follow from applying (2.12) and (2.13) to the action (2.9). The transformations (2.12) yield

$$q_\nu S^{\cdots XA^{1,2}Y\cdots}_{\cdots a\nu\;\;\; b\cdots}(\ldots,p,q,r,\ldots) \\ = S^{\cdots XY\cdots}_{\cdots ab\cdots}(\ldots,p,q+r,\ldots) - S^{\cdots XY\cdots}_{\cdots ab\cdots}(\ldots,p+q,r,\ldots). \tag{2.15}$$

The effect of the transformations (2.13) is most efficiently written in the five-component language of (2.14*a*) and (2.14*b*). Introducing a 5-momentum,

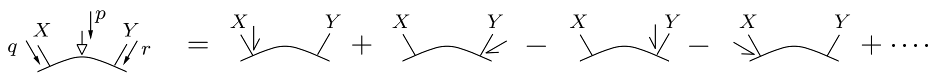
$$q_M = (q_\mu, 2), \tag{2.16}$$

allows us to write

$$q_N S^{\cdots XFY\cdots}_{\cdots aNb\cdots}(\ldots,p,q,r,\ldots) \\ = S^{\cdots X\overrightarrow{Y}\cdots}_{\cdots ab\cdots}(\ldots,p,q+r,\ldots) - S^{\cdots \overleftarrow{X}Y\cdots}_{\cdots ab\cdots}(\ldots,p+q,r,\ldots), \tag{2.17}$$

where $\overrightarrow{Y}$ and $\overleftarrow{X}$ are the opposite statistics partners of the fields Y and X. (For explicit expressions see [18, 19].) An identical expression to (2.17) exists for when the field F_N is replaced by $\bar{F}_N$.

The Ward identities (2.15) and (2.17) can be beautifully combined using the diagrammatics:



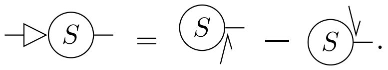
(2.18)

On the left-hand side, we contract a vertex with the momentum of the field which carries p. This field—which we will call the active field—can be either A^1_ρ, A^2_ρ, F_R or $\bar{F}_R$. In the first two cases, the triangle $\triangleright$ represents p_ρ whereas, in the latter two cases, it represents $p_R = (p_\rho, 2)$. (Given that we often sum over all possible fields, we can take the Feynman rule for $\triangleright$ in the C-sector to be null.) On the right-hand side of (2.18), we push the contracted momentum forward onto the field which directly follows the active field, in the counterclockwise sense, and pull back (with a minus sign) onto the field which directly precedes the active field. Since our diagrammatics is permutation symmetric, the struck field—which we will call the target field—can be either X, Y or any of the undrawn fields, as represented by the ellipsis.

Allowing the active field to strike another field necessarily involves a partial specification of the supertrace structure: it must be the case that the struck field either directly followed or preceded the active field. In turn, this means that the Feynman rule for particular choices of the active and target fields can be zero. For example, as trivially follows by multiplying together supermatrices, an F can follow, but never precede an A^1_μ, and so the pull back of an A^1_μ onto an F should be assigned a value of zero. The momentum routing follows in an obvious manner: for example, in the first diagram on the right-hand side, momenta $q + p$ and r now flow into the vertex. In the case that the active field is fermionic, the field pushed forward/pulled back onto is transformed into its opposite statistic partner, as above.

The half arrow which terminates the pushed forward/pulled back active field is of no significance and can go on either side of the active field line. It is necessary to keep the active field line—even though the active field is no longer part of the vertex—in order that we can unambiguously deduce flavour changes and momentum routing, without reference to the parent diagram.

We illustrate (2.18) by considering contracting $\triangleright$ into the Wilsonian the effective action two-point vertex:



(2.19)

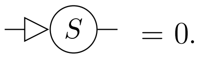
Given that $\triangleright$ is null in the C^i sector, the fields decorating the two-point vertex on the right-hand side can be either both A^is or both fermionic. In the former case, (2.19) reads

$$p_\mu S^{A^i A^i}_{\mu\ \nu}(p) = S^{A^i}_\nu(0) - S^{A^i}_\nu(0) = 0$$

where we note that $S^{A^i}_\nu$ is in fact zero by itself, as follows by both Lorentz invariance and gauge invariance. In the latter case, (2.19) reads

$$p_M S^{\bar{F}F}_{MN}(p) = \left[S^{C^2}(0) - S^{C^1}(0)\right]\delta_{N5},$$

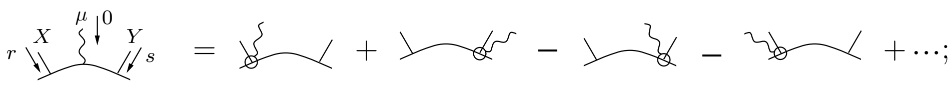
where we have used (2.14a) and have discarded contributions which go like $S^{A^i}_\nu(0)$. However, $S^{C^i}(0)$ must vanish. This follows from demanding that the minimum of the superhiggs potential is not shifted by quantum corrections [17]. Therefore,



(2.20)

2.3. *Taylor expansion of vertices*

For the formalism to be properly defined, it must be the case that all vertices are Taylor expandable to all orders in momenta [14–16]. Consider a vertex which is part of a complete diagram, decorated by some set of internal fields and by a single external A^1 (or A^2), which we denote by a wiggly line. The diagrammatic representation for the zeroth-order expansion in the momentum of the external field is all that is required for this paper [18, 19]:



(2.21)

note the similarity to (2.18).

The interpretation of the diagrammatics is as follows. In the first diagram on the right-hand side, the vertex is differentiated with respect to the momentum carried by the field X, whilst holding the momentum of the preceding field fixed (we assume for the time being that both X and the preceding field carry non-zero momentum). Of course, using our current diagrammatic notation, this latter field can be any of those which decorate the vertex, and so we sum over all possibilities. Thus, each cyclically ordered push-forward-like term has a partner, cyclically ordered pull-back-like term, such that the pair can be interpreted as

$$\left(\partial^r_\mu\big|_s - \partial^s_\mu\big|_r\right)\mathrm{Vertex}, \tag{2.22}$$

where r and s are momenta entering the vertex. In the case that $r = -s$, we can and will drop either the push-forward-like term or pull-back-like term, since the combination can be expressed as ∂^r_μ; we interpret the diagrammatic notation appropriately. If any of the fields decorating the vertex carry zero momentum (besides the explicitly drawn A^i), then they are transparent to this entire procedure. Thus, they are never differentiated and, if they precede a field which is, we must look to the first field carrying non-zero momentum to figure out which of the vertex's momenta is held constant.

3. Adding quarks

3.1. Massless quarks

The simplest way to try to incorporate quarks into the set-up is to embed them into the fundamental representation of $SU(N|N)$:

$$\Psi = \begin{pmatrix} \psi \\ \varphi \end{pmatrix},$$

where Ψ transforms under $SU(N|N)$, ψ is a physical quark field and φ is an unphysical, bosonic spinor (here and henceforth, we suppress spinor indices)[3]. Immediately, we can see that this embedding is inconsistent with (2.3): the supergauge invariant quark term,

$$\frac{1}{g^2}\mathrm{i}\bar{\Psi}\not{\nabla}\Psi,$$

does not satisfy no-$\mathcal{A}^0$ symmetry.

If, however, the fields Ψ transform as $R \otimes \bar{R}$, for some representation, R, then we can construct a no-$\mathcal{A}^0$ invariant representation simply because Ψ has zero 'charge' under no-$\mathcal{A}^0$. Thus, the strategy we pursue is to embed the quarks into fields which transform as a bifundamental of $SU(N|N)$. To achieve this, we first embed the up-like quarks (up, charm, top (suitably generalized for $N \neq 3$)) and down-like quarks (down, strange, bottom) into two tensor fields $(\psi_u)^i_j$ and $(\psi_d)^i_j$, where the superscript indices carry an $SU(N)$ colour symmetry and the subscript indices carry an (unphysical, gauged) $SU(N)$ flavour symmetry. In turn, ψ_u and ψ_d are now embedded into fields Ψ_u and Ψ_d which are valued in complexified $U(N|N)$:

$$\Psi_u = \begin{pmatrix} \varphi^1 & \psi_u \\ \varrho & \varphi^2 \end{pmatrix}, \qquad \Psi_d = \begin{pmatrix} \phi^1 & \psi_d \\ \rho & \phi^2 \end{pmatrix}. \tag{3.1}$$

Note that Ψ_u and Ψ_d are not Hermitian. Consequently, ϱ (ρ) are not related to the physical fields ψ_u (ψ_d) and, since they will be seen to come with wrong sign action, should be interpreted as unphysical degrees of freedom. These fields, together with $\phi^1, \phi^2, \varphi^1$ and φ^2 (the components of which are bosonic spinors), will be given a mass of order the cut-off. Under gauge transformations, Ψ_u and Ψ_d transform homogeneously:

$$\delta\Psi_u = -\mathrm{i}[\Psi_u, \Omega], \qquad \delta\Psi_d = -\mathrm{i}[\Psi_d, \Omega]. \tag{3.2}$$

The $SU(N|N)$ invariant quark kinetic term that we include in the Lagrangian is just

$$-\frac{\mathrm{i}}{g^2}(\operatorname{str}\bar{\Psi}_u \not{\nabla}\cdot\Psi_u + \operatorname{str}\bar{\Psi}_d \not{\nabla}\cdot\Psi_d), \tag{3.3}$$

where the minus sign compensates the sign buried in the supertrace, ensuring that the physical quark terms come with the correct overall sign. Note that we have not included any covariant higher derivatives; it is straightforward to demonstrate that the supergroup structure, alone, is sufficient to provide the necessary regularization in the quark sector by repeating the analysis of [28], but this time including the fields Ψ_i.

To show the types of terms that we must include in the action to give the unphysical fields a mass of order the cut-off but leave the quarks massless (for the time being), it is useful to construct the following projectors:

$$\sigma_+ \equiv \frac{1}{2}(1\!\!1 + \sigma) = \begin{pmatrix} 1\!\!1_N & 0 \\ 0 & 0 \end{pmatrix}, \qquad \sigma_- \equiv \frac{1}{2}(1\!\!1 - \sigma) = \begin{pmatrix} 0 & 0 \\ 0 & 1\!\!1_N \end{pmatrix}. \tag{3.4}$$

[3] If this scheme were to work, we would also have to introduce further unphysical fields to provide sufficient PV regularization.

With a slight abuse of notation, we can write $\varphi^1 = \sigma_+\Psi_u\sigma_+$, $\varphi^2 = \sigma_-\Psi_u\sigma_-$, $\varrho = \sigma_-\Psi_u\sigma_+$ and $\psi_u = \sigma_+\Psi_u\sigma_-$. We can lift these projectors to the symmetric phase by defining

$$\varsigma_\pm \equiv \tfrac{1}{2}(1\!\!1 \pm \mathcal{C}). \tag{3.5}$$

Thus, to give a mass to , e.g., ϱ in the broken phase, all we need to do is add to the Lagrangian the term

$$-\frac{1}{g^2}\Lambda\ \mathrm{str}(\bar{\Psi}_u\varsigma_-\Psi_u\varsigma_+).$$

Upon spontaneous symmetry breaking, this reduces to $-\Lambda/g^2\,\mathrm{tr}\,\bar{\varrho}\varrho$ (plus interaction terms).

Thus, in the broken phase, the only massless fields we are left with are ψ_i, A^1 and A^2. In the pure gauge case, we know from [28] that A^2 decouples from A^1: integrating out the heavy fields, the lowest dimension gauge-invariant effective interaction left between A^1 and A^2 is the square of the two field strengths (according to standard perturbative power counting with g in the usual place),

$$\Lambda^{-D}\,\mathrm{tr}(F^1)^2\,\mathrm{tr}(F^2)^2,$$

which is clearly irrelevant. Adding the quarks, however, we immediately see from (3.3) that these fields can combine with the unphysical gauge field to form a term of mass dimension D,

$$\alpha\,\mathrm{tr}(\bar{\psi}_u\psi_u + \bar{\psi}_d\psi_d)\,\not{A}^2.$$

To remove this term, and thus ensure that the unphysical fields decouple as the regularization scale is sent to infinity, we must tune α (see (2.6)) to zero at the end of a generic calculation. In fact, this tuning is perfectly natural from a non-perturbative perspective: the theory carried by A^2 is, as a consequence of its wrong sign action, not asymptotically free but instead trivial. Moreover, we can remove the need to perform any such tuning by modifying the Higgs sector to completely break the unphysical $SU(N)$; indeed, we will do precisely this when we adapt the formalism to give the quarks independent masses.

We conclude this section by discussing the additional (un)broken invariances which arise from the inclusion of the quarks. The up-like quarks supplement (2.12) with

$$\begin{aligned}
\delta\psi_u &= \mathrm{i}(\omega^1\psi_u - \psi_u\omega^2)\\
\delta\varrho &= \mathrm{i}(\omega^2\varrho - \varrho\omega^1)\\
\delta\varphi^1 &= \mathrm{i}\omega^1\cdot\varphi^1\\
\delta\varphi^2 &= \mathrm{i}\omega^2\cdot\varphi^2
\end{aligned} \tag{3.6}$$

and (2.13) with

$$\begin{aligned}
\delta\psi_u &= \mathrm{i}(\tau\varphi^2 - \varphi^1\tau)\\
\delta\varrho &= \mathrm{i}(\bar{\tau}\varphi^1 - \varphi^2\bar{\tau})\\
\delta\varphi^1 &= \mathrm{i}(\tau\varrho - \psi_u\bar{\tau})\\
\delta\varphi^2 &= \mathrm{i}(\bar{\tau}\psi_u - \varrho\tau)
\end{aligned} \tag{3.7}$$

(similarly for the down-like quarks). Note that the quark field ψ_u is not protected from field strength renormalization. However, the transformations (3.7) do enforce that all components of Ψ_u have the same field strength renormalization (likewise Ψ_d). The unbroken transformation for ψ_u given by (3.6) confirms our interpretation that the physical colour symmetry is carried by A^1, whereas the unphysical flavour symmetry is carried by A^2.

3.2. Massive quarks

If we only needed to give all the up-like quarks one mass and all the down-like quarks one mass, then we could simply add a mass term to the Lagrangian, using the fields that we have already:

$$-\frac{1}{g^2}[m_u \,\mathrm{str}\, \bar{\Psi}_u \varsigma_+ \Psi_u \varsigma_- + m_d \,\mathrm{str}\, \bar{\Psi}_d \varsigma_+ \Psi_d \varsigma_-],$$

where we have used (3.5). Of course, to give all the quarks different masses, we will have to break the unphysical, gauged flavour symmetry. To do this, we introduce two new dimensionless superscalars, $\mathcal{C}_u$ and $\mathcal{C}_d$ which, like $\mathcal{C}$, lie in the adjoint of $U(N|N)$ and transform homogeneously:

$$\delta\mathcal{C}_u = -\mathrm{i}[\mathcal{C}_u, \Omega], \qquad \delta\mathcal{C}_d = -\mathrm{i}[\mathcal{C}_d, \Omega].$$

We choose the vevs of $\mathcal{C}_u$ and $\mathcal{C}_d$ to be

$$\langle\mathcal{C}_u\rangle = \begin{pmatrix} 1\!\!1_N & 0 \\ 0 & -\sigma_u \end{pmatrix}, \qquad \langle\mathcal{C}_d\rangle = \begin{pmatrix} 1\!\!1_N & 0 \\ 0 & -\tilde{\sigma}_d \end{pmatrix}, \tag{3.8}$$

where $\sigma_u = \mathrm{diag}(1, m_c/m_u, m_t/m_u)$ and, given the unitary matrix, $U, U^\dagger\tilde{\sigma}_d U = \sigma_d = \mathrm{diag}(1, m_s/m_d, m_b/m_d)$ (with an obvious generalization for arbitrary values of N). The *vev* of $\mathcal{C}_u$ breaks the unphysical $SU(N)$ down to $U(1)^{N-1}$. We could choose the vev of $\mathcal{C}_d$ to be diagonal. However, in this case we would be forced to tune the couplings of the residual $U(1)$ s (which we note are, on account of their wrong sign action, asymptotically free and not trivial) to zero. Consequently, we might as well choose the vev of $\mathcal{C}_d$ such that in combination with the vev of $\mathcal{C}_u$ the unphysical $SU(N)$ is completely broken. Thus, we implicitly assume U to be such that amongst the generators broken by $\langle\mathcal{C}_d\rangle$ are those which are not broken by $\langle\mathcal{C}_u\rangle$. The vevs of $\mathcal{C}_u$ and $\mathcal{C}_d$ (unlike that of $\mathcal{C}$) are not protected from quantum corrections, which of course corresponds to the renormalization of the quark masses. The broken Ward identities (whose modification due to the breaking of the unphysical $SU(N)$ we will discuss shortly) protect the components of $\mathcal{C}_{u,d}$ from field strength renormalization, in the broken phase.

With the introduction of $\mathcal{C}_u$ and $\mathcal{C}_d$, there is no requirement to retain $\mathcal{C}$. However, the following exposition is made simpler if we keep $\mathcal{C}$ and so we do so, noting that such considerations are anyway irrelevant from the point of view of the diagrammatic form of the flow equation. We will not give an explicit realization of the symmetry breaking potential $V(\mathcal{C}, \mathcal{C}_u, \mathcal{C}_d)$ which yields (2.5) and (3.8), since we are free to work with any potential which satisfies the following requirements. First (in unitarity gauge), all Goldstone bosons are eaten by the various components of $\mathcal{A}_\mu$ which acquire mass (i.e., the potential must not possess any accidental symmetries: the largest continuous symmetry group is just $SU(N|N)$). Secondly, the remaining (Higgs) components of $\mathcal{C}$, $\mathcal{C}_u$ and $\mathcal{C}_d$, are given a mass of order the cut-off.

For non-degenerate masses, it is useful (and always possible) to construct the set of N projectors which live in the bottom right block of a supermatrix:

$$\begin{aligned} P_1 &= \mathrm{diag}(0_N, 1, 0, 0, \ldots) \\ P_2 &= \mathrm{diag}(0_N, 0, 1, 0, \ldots) \\ &\vdots \end{aligned} \tag{3.9}$$

To see this, simply note that, e.g.,

$$\frac{(\langle\mathcal{C}_u\rangle - \langle\mathcal{C}\rangle)(\langle\mathcal{C}_u\rangle - m_c/m_u\,\langle\mathcal{C}\rangle)}{(1 - m_t/m_u)(m_c/m_u - m_t/m_u)} = \mathrm{diag}(0, 0, 0, 0, 0, 1).$$

We can lift P_i to the symmetric phase by introducing the non-degenerate (running) parameters a_i and defining

$$\mathcal{P}_j = \prod_{i\neq j} \frac{\mathcal{C}_u - a_i\mathcal{C}}{a_i - a_j}. \tag{3.10}$$

In the broken phase (which recall that, by construction, we are actually always in), we identify a_i with the elements of σ_u. Note that the $\mathcal{P}_j$ gauge transform homogeneously.

The quarks' mass term can be taken to be

$$\frac{1}{g^2}[m_u\,\mathrm{str}(\mathcal{C}_u\varsigma_-\bar{\Psi}_u\varsigma_+\Psi_u\varsigma_-) + m_d\,\mathrm{str}(\mathcal{C}_d\varsigma_-\bar{\Psi}_d\varsigma_+\Psi_d\varsigma_-)], \tag{3.11}$$

where it is understood from now on that we have rotated the down-like quark fields to the mass basis (this is exactly analogous to the introduction of the CKM matrix in the standard model). The remaining components of $\Psi_{u,d}$ are given masses of order the cut-off. Note that we have included $\varsigma_\pm$ in (3.11) purely for convenience, to ensure that the masses of the components of the fields ϕ^2 and φ^2 (see (3.1)) do not pick up contributions from the quarks' mass matrices.

Neglecting the covariant higher derivative regularization, the kinetic terms for $\mathcal{C}_u$ and $\mathcal{C}_d$ take the form

$$\frac{1}{2g^2}\,\mathrm{str}[(\nabla_\mu\cdot\mathcal{C}_u)^2 + (\nabla_\mu\cdot\mathcal{C}_d)^2].$$

Note that this term provides differing contributions to the masses of the various components of B. Specifically, B decomposes into columns, with each column receiving a different mass. This is precisely what we would expect from the unbroken gauge transformations, as we now discuss.

An immediate effect of breaking the unphysical $SU(N)$ is that the relationships (2.12) and (2.13), (3.6) and (3.7) (which we supplement by those appropriate for $\mathcal{C}_{u,d}$) decompose. The only relationships which are completely unaffected are those involving just ω_1, i.e., the unbroken relationships for A^1_μ, C^1, $C^1_{u,d}$, φ^1 and ϕ^1.

The relationships involving just ω_2 are completely broken. This means that the independent components of each of the bottom right block fields are no longer related by unbroken gauge transformations and so can be expected to propagate separately.

In the fermionic sectors, the previously unbroken gauge transformations involve both ω_1 and ω_2 , e.g., $\delta B_\mu = -\mathrm{i}(B_\mu\omega^2 - \omega^1 B_\mu)$. Now, however, the ω_2 part is completely broken. In matrix language, the surviving unbroken transformation involving ω_1 mixes up elements of each column with elements of the *same* column. Consequently, upon the breaking of the unphysical $SU(N)$, B_μ decomposes into N 'flavours', $B_{\mu a}$, corresponding to the N columns, with unbroken transformation law

$$(\delta B_{\mu a})^i = \mathrm{i}(\omega^1)^i_{\ j}(B_{\mu a})^j.$$

This is precisely what we want: in the quark sector, colour remains a good symmetry and the unphysical, gauged flavour symmetry is completely broken.

With the above decomposition of many of our fields, we must adapt our expansion of the action in terms of fields (2.9) by appropriately expanding the set of fields represented by X and Y. To maintain the supermatrix structure, we should ensure that the new fields are still embedded in supermatrices. For example, the field $B_{\mu a}$ should be in the appropriate column of the top-right block of a supermatrix, with all other elements set to zero. Equivalently, we can project the field $B_{\mu a}$ out by using P_i of (3.9).

At first sight, the breaking of the unphysical $SU(N)$ considerably complicates the Ward identities. However, we can anticipate from [23–25] that we can and should hide these complications in the diagrammatics. This will be made particularly straightforward if we now sum over the flavours of the target fields in (2.18). This helps for the following reason. Consider (2.15) where the target fields are fermionic. We know that the fermionic fields decompose by column and so the right-hand side of (2.15) will contain a sum of terms such that, if the unphysical $SU(N)$ were restored, these terms could be combined back into block supermatrix components.

We conclude this section by giving the renormalization conditions for the quarks:

$$S = \frac{1}{g^2}\int \mathrm{d}^D x\ \mathrm{tr}[\bar\psi_u(i\,\nabla\!\!\!\!/^{\,1} + \sigma_u)\psi_u + \bar\psi_d(i\,\nabla\!\!\!\!/^{\,1} + \sigma_d)\psi_d] + \cdots,$$

where the ellipsis represents all other operators contributing to the effective action.

4. A flow equation for QCD

4.1. Review of $SU(N)$ Yang–Mills

4.1.1. Set-up. We begin by describing the flow equation used for pure Yang–Mills [19], the basic form of which is

$$-\Lambda\partial_\Lambda S = a_0[S, \Sigma_g] - a_1[\Sigma_g], \tag{4.1}$$

where $\Sigma_g \equiv g^2 S - 2\hat S$ ($\hat S$, we recall, being the seed action).[4] On the right-hand side of the flow equation is the bilinear functional, $a_0[S, \Sigma_g]$, which generates classical corrections and the functional $a_1[\Sigma_g]$ which generates quantum corrections. These terms are given by

$$a_0[S, \Sigma_g] = \frac{1}{2}\frac{\delta S}{\delta \mathcal{A}_\mu}\{\dot\Delta^{\mathcal{A}\mathcal{A}}\}\frac{\delta \Sigma_g}{\delta \mathcal{A}_\mu} + \frac{1}{2}\frac{\delta S}{\delta \mathcal{C}}\{\dot\Delta^{\mathcal{C}\mathcal{C}}\}\frac{\delta \Sigma_g}{\delta \mathcal{C}}, \tag{4.2}$$

$$a_1[\Sigma_g] = \frac{1}{2}\frac{\delta}{\delta \mathcal{A}_\mu}\{\dot\Delta^{\mathcal{A}\mathcal{A}}\}\frac{\delta \Sigma_g}{\delta \mathcal{A}_\mu} + \frac{1}{2}\frac{\delta}{\delta \mathcal{C}}\{\dot\Delta^{\mathcal{C}\mathcal{C}}\}\frac{\delta \Sigma_g}{\delta \mathcal{C}}, \tag{4.3}$$

where $\dot\Delta$ represent the ERG kernels and the notation $\{\dot\Delta\}$ denotes their covariantization [14, 15].

The natural definitions of functional derivatives of $SU(N|N)$ matrices are used [16, 17, 28]:

$$\frac{\delta}{\delta \mathcal{C}} \equiv \begin{pmatrix} \delta/\delta C^1 & -\delta/\delta \bar D \\ \delta/\delta D & -\delta/\delta C^2 \end{pmatrix}, \tag{4.4}$$

and from (2.2) [17, 28]

$$\frac{\delta}{\delta \mathcal{A}_\mu} \equiv 2T_A \frac{\delta}{\delta \mathcal{A}_{A\mu}} + \frac{\sigma}{2N}\frac{\delta}{\delta \mathcal{A}^0_\mu}. \tag{4.5}$$

The wonderful simplicity of (4.2) and (4.3) arises from the realization that the fine detail of the flow equation (which, as we will see, does not affect universal quantities anyway) can be buried in the definition of the covariantization. Nonetheless, for the purpose of transparently generalizing to QCD, we now discuss the covariantization in some detail. The primary ingredient is the supercovariantization [17] of the kernel W, $\{W\}_{\mathcal{A}}$. This is defined according to

$$u\,\{W\}_{\mathcal{A}} v = \sum_{m,n=0}^{\infty}\int_{x_1,\ldots,x_n;y_1,\ldots,y_m;x,y} W_{\mu_1\cdots\mu_n,\nu_1\cdots\nu_m}(x_1,\ldots,x_n;y_1,\ldots,y_m;x,y)$$
$$\mathrm{str}[u(x)\mathcal{A}_{\mu_1}(x_1)\cdots\mathcal{A}_{\mu_n}(x_n)v(y)\mathcal{A}_{\nu_1}(y_1)\cdots\mathcal{A}_{\nu_m}(y_m)], \tag{4.6}$$

[4] Note that we use Σ_g instead of the Σ_1 of (1.2), on account of g being scaled out of the covariant derivative.

where u and v are the supermatrix representations transforming homogeneously as in (2.4) and where, without loss of generality, we may insist that $\{W\}_{\mathcal{A}}$ satisfies $u\,\{W\}_{\mathcal{A}}v \equiv v\,\{W\}_{\mathcal{A}}u$. For simplicity's sake, we have chosen (4.6) to contain only a single supertrace. (In the diagrammatic form of the flow equation, such details make no difference.) The $m = n = 0$ term is just the original kernel, i.e.,

$$W_{,}(;\,;x,y) \equiv W_{xy}. \tag{4.7}$$

The requirement that (4.6) is supergauge invariant enforces a set of Ward identities on the vertices $W_{\mu_1\cdots\mu_n,\nu_1\cdots\nu_m}$ which we describe later. The no-$\mathcal{A}^0$ symmetry is obeyed by requiring the coincident line identities [15]. These identities are equivalent to the requirement that the gauge fields all act by commutation [16], ensuring that the no-$\mathcal{A}^0$ part of (2.3) is satisfied. A consequence of the coincident line identities, which also trivially follows from the representation of (4.6) in terms of commutators, is that if $v(y) = 1\!\!1 g(y)$ for all y, i.e., is in the scalar representation of the gauge group, then the covariantization collapses to

$$u\,\{W\}_{\mathcal{A}}v = (\mathrm{str}\,u)\cdot W\cdot g, \tag{4.8}$$

where we define

$$f\cdot W\cdot g = \int_{x,y} f(x)W_{xy}g(y) = \int_x f(x)W(-\partial^2/\Lambda^2)g(x)$$

which holds for any momentum space kernel $W(p^2/\Lambda^2)$ and functions of spacetime f, g, using

$$W_{xy} = W(-\partial^2/\Lambda^2)\delta(x-y) = \int \frac{\mathrm{d}^D p}{(2\pi)^D} W(p^2/\Lambda^2)\,\mathrm{e}^{\mathrm{i}p\cdot(x-y)}.$$

At this point, it is instructive to recall the demonstration of the $SU(N|N)$ invariance of the flow equation, assuming that the covariantizations $\{\dot\Delta\}$ are just of the type (4.6) [17].

Under (2.4), the $\mathcal{C}$ functional derivative transforms homogeneously:

$$\delta\left(\frac{\delta}{\delta\mathcal{C}}\right) = -\mathrm{i}\left[\frac{\delta}{\delta\mathcal{C}}, \Omega\right], \tag{4.9}$$

and thus by (4.6) the corresponding terms in (4.2) and (4.3) are invariant. The $\mathcal{A}$ functional derivative, however, transforms as [17]

$$\delta\left(\frac{\delta}{\delta\mathcal{A}_\mu}\right) = -\mathrm{i}\left[\frac{\delta}{\delta\mathcal{A}_\mu}, \Omega\right] + \frac{\mathrm{i}1\!\!1}{2N}\,\mathrm{tr}\left[\frac{\delta}{\delta\mathcal{A}_\mu}, \Omega\right]. \tag{4.10}$$

The correction is there because (4.5) is traceless, which in turn is a consequence of the supertracelessness of (2.1). The fact that $\delta/\delta\mathcal{A}$ does not transform homogeneously means that supergauge invariance is destroyed unless the correction term vanishes for other reasons.

Here, no-$\mathcal{A}^0$ symmetry comes to the rescue. Using the invariance of (4.6) for homogeneously transforming u and v, and the invariance of S and $\hat{S}$, we have by (4.10) and (4.8) that the $\mathcal{A}$ term in (4.2) transforms to

$$\delta\left(\frac{\delta S}{\delta\mathcal{A}_\mu}\{\dot\Delta^{\mathcal{A}\mathcal{A}}\}\frac{\delta\Sigma_g}{\delta\mathcal{A}_\mu}\right) = \frac{\mathrm{i}}{2N}\,\mathrm{tr}\left[\frac{\delta S}{\delta\mathcal{A}_\mu}, \Omega\right]\cdot\dot\Delta^{\mathcal{A}\mathcal{A}}\cdot\,\mathrm{str}\,\frac{\delta\Sigma_g}{\delta\mathcal{A}_\mu} + (S\leftrightarrow\Sigma_g), \tag{4.11}$$

where $S\leftrightarrow\Sigma_g$ stands for the same term with S and Σ_g interchanged. But by (4.5) and no-$\mathcal{A}^0$ symmetry,

$$\mathrm{str}\,\frac{\delta\Sigma_g}{\delta\mathcal{A}_\mu} = \frac{\delta\Sigma_g}{\delta\mathcal{A}^0_\mu} = 0$$

(similarly for S), and thus the tree level terms are invariant under (2.3) and (2.4). Likewise, the quantum terms in (4.3) are invariant and this completes the proof that, for the covariantizations of the form (4.6), the flow equation is both supergauge and no-$\mathcal{A}^0$ invariant.

As it stands, the covariantization (4.6) is not general enough for our purposes: we require the broken phase fields to come with their own kernels. The first part of the solution to this [17] is to define a new covariantization

$$u\,\{W\}_{\mathcal{AC}}v = u\,\{W\}_{\mathcal{A}}v - \tfrac{1}{4}[\mathcal{C}, u]\,\{W_m\}_{\mathcal{A}}[\mathcal{C}, v]. \tag{4.12}$$

The $\mathcal{C}$ commutator terms are introduced to allow a difference between A and B kernels, and C and D kernels, in the broken phase. They do this because at the level of two-point flow equations $\mathcal{C}$ is replaced by σ in (4.12), and σ (anti)commutes with the (fermionic) bosonic elements of the algebra. Thus, extracting the broken phase two-point, classical flow equations from (4.2), we find that the A^i kernels are given by $\dot\Delta^{\mathcal{AA}}$, the C^i kernels by $\dot\Delta^{\mathcal{CC}}$, but the B kernel is $\dot\Delta^{\mathcal{AA}} + \dot\Delta^{\mathcal{AA}}_m$ and the D kernel is $\dot\Delta^{\mathcal{CC}} + \dot\Delta^{\mathcal{CC}}_m$ [17]. The B and D kernels can be combined (cf (2.14a) and (2.14b)):

$$\dot\Delta^{F\bar F}_{MN}(p) = \begin{pmatrix} \dot\Delta^{B\bar B}_p \delta_{\mu\nu} & 0 \\ 0 & -\dot\Delta^{D\bar D}_p \end{pmatrix}.$$

As must be the case, the extra term in (4.12) is consistent with both supergauge and no-$\mathcal{A}^0$ invariance. If u and v transform homogeneously, then so do $[\mathcal{C}, u]$ and $[\mathcal{C}, v]$. Correction terms proportional to the identity (cf (4.10)) are killed by the commutator structure; this structure also ensures no-$\mathcal{A}^0$ invariance. Note that (4.8) holds for the extended covariantization.

We are still not quite done: it is convenient to generalize the covariantization yet further. In particular, since the physical coupling, g, and the unphysical coupling, g_2, renormalize differently [17, 19], it is useful to furnish A^1 and A^2 with different kernels (there is no need to do this for C^1 and C^2 [19]). To construct a term which does this, we cannot use a commutator, since $[\sigma, \delta/\delta A^i] = 0$. The simplest solution is to define in (4.2) and (4.3)

$$u\{\dot\Delta^{\mathcal{AA}}\}v \equiv u\,\{\dot\Delta^{\mathcal{AA}}\}_{\mathcal{AC}}v + u\,\{\dot\Delta^{\mathcal{AA}}_\sigma\}_{\mathcal{A}}\mathcal{P}(v) + \mathcal{P}(u)\,\{\dot\Delta^{\mathcal{AA}}_\sigma\}_{\mathcal{A}}v, \tag{4.13}$$

where

$$8N\mathcal{P}(X) = \{\mathcal{C}, X\}\,\mathrm{str}\,\mathcal{C} - 2\mathcal{C}\,\mathrm{str}\,\mathcal{C}X. \tag{4.14}$$

$\mathcal{P}(X)$ has the following properties which ensure both no-$\mathcal{A}^0$ invariance and supergauge invariance:

$$\mathrm{str}\,\mathcal{P}(X) = 0, \tag{4.15a}$$
$$\mathcal{P}(\mathbb{1}) = 0. \tag{4.15b}$$

In the broken phase, we see that

$$\mathcal{P}\left(\frac{\delta}{\delta\mathcal{A}_\mu}\right) = \frac{1}{2}\sigma\frac{\delta}{\delta\mathcal{A}_\mu} + \cdots,$$

where we have noted from (4.5) that $\mathrm{str}\,\sigma\delta/\delta\mathcal{A}_\mu = 0$ and the ellipsis includes terms with additional fields. Thus, a minus sign is introduced in the A^2 sector, compared to the A^1 sector, and we find that $\dot\Delta^{A^1A^1} = \dot\Delta^{AA} + \dot\Delta^{AA}_\sigma$, $\dot\Delta^{A^2A^2} = \dot\Delta^{AA} - \dot\Delta^{AA}_\sigma$ [19].

We conclude this section by commenting on the Ward identities satisfied by the vertices of the covariantized kernels. Throughout this section, we have used the fact that u and v in, e.g., (4.13) are functional derivatives with respect to, say, Z^1 and Z^2 to label the kernels, namely, $\dot\Delta^{Z^1Z^2}$. The diagrammatic form of the Ward identities (2.18) holds for the vertices of the kernels, also, so long as two of the target fields are identified with the ends of the kernels, i.e., with Z^1 and Z^2 [18, 19].

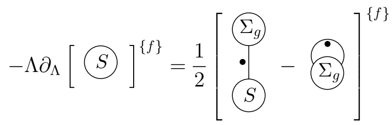


Figure 1. The diagrammatic form of the flow equation.

4.1.2. Diagrammatics. As mentioned in the introduction, the most useful representation of the flow equation is a diagrammatic one, which is shown in figure 1 [18, 19, 23, 25].

The term on the left-hand side generates the flow of all cyclically independent Wilsonian effective action vertex coefficient functions which correspond to the set of broken phase fields $\{f\}$.

The objects on the right-hand side of figure 1 have two different types of component. The lobes represent vertices of action functionals. The object attaching to the various lobes, —•—, is the sum over vertices of the covariantized ERG kernels [15, 17] and, like the action vertices, can be decorated by fields belonging to $\{f\}$. The fields of the action vertex (vertices) to which the vertices of the kernels attach act as labels for the ERG kernels. We henceforth loosely refer to both individual and summed over vertices of the kernels simply as a kernel. The dumbbell-like term corresponds to the classical term, a_0, whereas the padlock-like diagram corresponds to the quantum term, a_1.[5] The rule for decorating the classical and quantum terms is simple: the set of fields, $\{f\}$, are distributed in all independent ways between the component objects of each diagram.

Embedded within the diagrammatic rules is a prescription for evaluating the group theory factors. Suppose that we wish to focus on the flow of a particular vertex coefficient function which, necessarily, has a unique supertrace structure. For example, we might be interested in just the $S^{C^1C^1}$ component of (2.11). On the right-hand side of the flow equation, we must focus on the components of each diagram with precisely the same supertrace structure as the left-hand side, noting that the kernel, like the vertices, has multi-supertrace contributions. In this more explicit diagrammatic picture, the kernel is to be considered a double-sided object (for more details see [18, 19]). Thus, whilst the dumbbell-like term of figure 1 has at least one associated supertrace, the next diagram has at least two, on account of the loop (this is strictly true only in the case that kernel attaches to fields on the same supertrace). If a closed circuit formed by a kernel is devoid of fields then it contributes a group theory factor, depending on the flavours of the fields to which the kernel forming the loop attaches. This is most easily appreciated by noting that $\mathrm{str}\,\sigma_\pm = \pm N$ (see (3.4)). In the counterclockwise sense, a σ_+ can always be inserted for free after an A^1, C^1 or $\bar{F}$, whereas a σ_- can always be inserted for free after an A^2, C^2 or F.

The above prescription for evaluating the group theory factors receives $1/N$ corrections in the A^1 and A^2 sectors, as a consequence of the $SU(N)$ completeness relation [15]. If a kernel attaches to an A^1 or A^2, it comprises a direct attachment and an indirect attachment. In the former case, one supertrace associated with some vertex coefficient function is 'broken open' by an end of a kernel: the fields on this supertrace and the single supertrace component of the kernel are on the same circuit. In the latter case, the kernel does not break anything open and so the two sides of the kernel pinch together at the end associated with the indirect attachment. This is illustrated in figure 2, for more detail see [18, 19, 26].

[5] There is an additional, improperly regularized term generated by the flow equation which has been removed by a suitable constraint on the covariantization [14, 17, 19, 25].

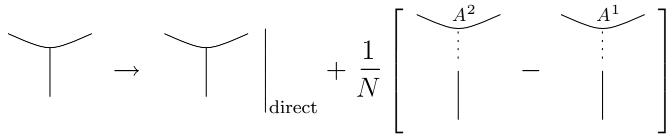


Figure 2. The $1/N$ corrections to the group theory factors.

We can thus consider the diagram on the left-hand side as having been unpackaged, to give the terms on the right-hand side. The dotted lines in the diagrams with indirect attachments serve to remind us where the loose end of the kernel attaches in the parent diagram.

4.2. Adding quarks

The game now is easy. We add to the flow equation classical and quantum terms for the fields Ψ_u, Ψ_d and modify the superhiggs sector. For all fields, we ensure that there is sufficient freedom in the covariantization to allow enough different kernels for the broken phase fields.

Though the details will ultimately be hidden in the diagrammatic form of the flow equation, we will give examples of choices we can make for the covariantization. Just as in the pure Yang–Mills case, the kernels of the propagating fields may turn out to be linear combinations of those which appear in the flow equation. Though we will not explicitly perform this change of basis, it is instructive to see how we can, if we so desire, construct the covariantization so as to make this procedure as easy as possible.

As in the pure Yang–Mills case, the starting point for constructing the covariantization is (4.6). To this we now add additional terms which reflect the complete breaking of the unphysical $SU(N)$ gauge symmetry. Knowing that B_μ decomposes into N flavours, it makes sense to add to the covariantization a term of the form

$$-\frac{1}{4}\sum_{j=1}^{N}\left([\mathcal{P}_j,[\mathcal{C},u]]\left\{\dot{\Delta}_j^{\mathcal{A}\mathcal{A}}\right\}_{\mathcal{A}}[\mathcal{P}_j,[\mathcal{C},v]]\right),$$

where we have used (3.10). (As usual, the overall factor is merely a matter of convention.) The presence of the $\mathcal{C}$ commutators is purely for convenience. In the broken phase, they project onto the block off-diagonal components of u and v, which ensures that, at the two-point level, the above term does not interfere with the flow of the components of the field A^2_μ. This makes it easier to extract the kernels of the propagating fields in terms of the kernels in the flow equation.

The gauge field A^2_μ has N^2-1 independent components. Since we can use the same kernel for a field and its Hermitian conjugate, we require a total of $N(N+1)/2-1$ kernels. So, for the components of A^2_μ, we add to the covariantization a term of the form

$$\sum_{j=1}^{N(N+1)/2-1}\left[u\left\{\dot{\Delta}_{j+N}^{\mathcal{A}\mathcal{A}}\right\}_{\mathcal{A}}\mathcal{P}'_j(\mathcal{P}(v))+\mathcal{P}'_j(\mathcal{P}(u))\left\{\dot{\Delta}_{j+N}^{\mathcal{A}\mathcal{A}}\right\}_{\mathcal{A}}v\right],\tag{4.16}$$

where

$$\mathcal{P}'_j(X)=(Y_jXZ_j+Z_jXY_j)\,\mathrm{str}\,Y_jZ_j-Y_jZ_j\,\mathrm{str}\,Y_jXZ_j-Z_jY_j\,\mathrm{str}\,Z_jXY_j,$$

and Y_j and Z_j contain linear combinations of the symmetric phase versions of the projectors defined by (3.10) in their bottom right block (all other elements being zero). There are clearly many different choices we can take for Y_j and Z_j. An example for $N>2$ would be to set

$Y_j = Z_j$ and choose the first $N(N-1)/2$ Y_j to be such that, in the broken phase, they reduce to the ${}^N C_2$ independent combinations of projectors of the form $Y_j = P_k + P_{l\neq k}$. Then we can take the remaining Y_j to be any $N-1$ of the N P_i. Note that $\mathcal{P}'$ satisfies the conditions (4.15a) and (4.15b) and thus does not spoil either the supergauge or no-$\mathcal{A}^0$ invariance of the flow equation. The appearance of $\mathcal{P}$ (see (4.14)) in (4.16) is again for convenience ensuring that, at the two-point level, the kernels of (4.16) do not appear in the flow of the components of B_μ.

Finally, then, a suitable choice for the covariantization of $\dot{\Delta}^{\mathcal{A}\mathcal{A}}$ is

$$u\{\dot{\Delta}^{\mathcal{A}\mathcal{A}}\}v \equiv u\left\{\dot{\Delta}^{\mathcal{A}\mathcal{A}}\right\}_{\mathcal{A}} v - \frac{1}{4}\sum_{j=1}^{N}\left([\mathcal{P}_j,[\mathcal{C},u]]\left\{\dot{\Delta}_j^{\mathcal{A}\mathcal{A}}\right\}_{\mathcal{A}}[\mathcal{P}_j,[\mathcal{C},v]]\right)$$
$$+\sum_{j=1}^{N(N+1)/2-1}\left[u\left\{\dot{\Delta}_{j+N}^{\mathcal{A}\mathcal{A}}\right\}_{\mathcal{A}}\mathcal{P}_j'(\mathcal{P}(v)) + \mathcal{P}_j'(\mathcal{P}(u))\left\{\dot{\Delta}_{j+N}^{\mathcal{A}\mathcal{A}}\right\}_{\mathcal{A}} v\right].$$

The modifications to the covariantization in the superhiggs sector are almost identical; the only real difference is that, since the superscalars are not supertraceless, there are more propagating degrees of freedom than in the $\mathcal{A}_\mu$ sector, and so we must introduce additional kernels to take account of this.

The inclusion of quarks follows a similar pattern. The contributions of the up-like quarks to the flow equation are the standard ones for spinor fields [33][6], with the contribution to the classical term given by

$$\frac{1}{2}\left(\frac{\delta S}{\delta\bar{\Psi}_u}\{\dot{\Delta}^{\bar{\Psi}_u\Psi_u}\}\frac{\delta\Sigma_g}{\delta\Psi_u} + \frac{\delta S}{\delta\Psi_u}\{\dot{\Delta}^{\bar{\Psi}_u\Psi_u}\}\frac{\delta\Sigma_g}{\delta\bar{\Psi}_u}\right),$$

and the contribution to quantum term given by

$$\frac{1}{2}\left(\frac{\delta}{\delta\Psi_u}\{\dot{\Delta}^{\bar{\Psi}_u\Psi_u}\}\frac{\delta\Sigma_g}{\delta\bar{\Psi}_u} + \frac{\delta}{\delta\bar{\Psi}_u}\{\dot{\Delta}^{\bar{\Psi}_u\Psi_u}\}\frac{\delta\Sigma_g}{\delta\Psi_u}\right),$$

where we have suppressed spinor indices and functional derivatives with respect to Ψ are defined as for any other unconstrained (i.e., not supertraceless) superfield (see (4.4)).

The covariantization is chosen to be

$$u\{\dot{\Delta}^{\bar{\Psi}_u\Psi_u}\}v = u\left\{\dot{\Delta}^{\bar{\Psi}_u\Psi_u}\right\}_{\mathcal{A}} v + \sum_{j=1}^{N}[\mathcal{P}_j,\varpi_-(u)]\{\dot{\Delta}_j^{\bar{\Psi}_u\Psi_u}\}_{\mathcal{A}}[\mathcal{P}_j,\varpi_+(v)] + \cdots. \tag{4.17}$$

The first term on the right-hand side is the usual contribution involving just the supercovariantization. In the second term we have introduced, for convenience, the objects

$$\varpi_\pm(X) = \tfrac{1}{8}\left([\mathcal{C},[\mathcal{C},X]] \pm 2[\mathcal{C},X]\right)$$

defined so that, in the broken phase, they reduce to $\sigma_+ X\sigma_-$ and $\sigma_- X\sigma_+$ (plus interaction terms). Consequently, at the two-point level, the second term on the right-hand side of (4.17) contributes only to the flow of the physical quarks. The ellipsis represents additional terms which provide kernels for the rest of the propagating fields embedded in Ψ_u.

4.3. Diagrammatics for QCD

4.3.1. The exact flow equation. The beauty of the diagrammatic form of the flow equation given in figure 1 is that it can be directly generalized from $SU(N)$ Yang–Mills to QCD: all we

[6] However, compared to QED, there is no need to explicitly take care of the anticommuting nature of the quark fields, since this is automatically taken care of by the embedding into the supergroup.

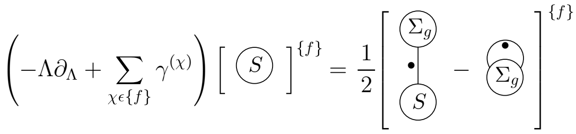


Figure 3. The diagrammatic form of the flow equation.

need to do is to extend the set of broken phase fields which can decorate the diagrams both as internal and external fields. A consequence of this is that the prescription for extracting the group theory factors receives further corrections, which follow from inserting the appropriate projectors to go from derivatives with respect to supermatrix block fields to derivatives with respect to their appropriate components.

In fact, it is useful to employ a slightly different flow equation, in which we work directly with renormalized fields [31–33] (recall that the components of all fields bar Ψ_u and Ψ_d are protected from field strength renormalization). This flow equation is given in figure 3. It is not the result of scaling the wavefunction renormalizations out of the version of (4.1) appropriate to QCD but is a perfectly valid flow equation, nonetheless [31, 32]. This is a manifestation of the tremendous freedom we have in constructing flow equations, encapsulated in (1.1).

The term $\sum_{\chi\epsilon\{f\}}\gamma^{(\chi)}$ explicitly takes account of the anomalous dimensions of the fields which suffer field strength renormalization. The field χ belongs to the set of fields $\{f\}$ and the notation $\gamma^{(\chi)}$ just stands for the anomalous dimension of the field χ (which is zero for all but the components of $\Psi_{u,d}$).

4.3.2. Perturbative diagrammatics. In the perturbative domain, we have the following weak coupling expansions [15, 17–19]. The Wilsonian effective action is given by

$$S = \sum_{i=0}^{\infty} g^{2(i-1)} S_i = \frac{1}{g^2} S_0 + S_1 + \cdots, \tag{4.18}$$

where S_0 is the classical effective action and $S_{i>0}$ is the ith-loop corrections. The seed action has a similar expansion:

$$\hat{S} = \sum_{i=0}^{\infty} g^{2i} \hat{S}_i, \tag{4.19}$$

and the β-function and anomalous dimensions are defined as usual:

$$\beta \equiv \Lambda\partial_\Lambda g = \sum_{i=1}^{\infty} g^{2i+1}\beta_i, \tag{4.20}$$

$$\gamma^{(\chi)} \equiv \Lambda\partial_\Lambda \ln Z^{(\chi)} = \sum_{i=1}^{\infty} g^{2i}\gamma_i^{(\chi)}, \tag{4.21}$$

where $Z^{(\chi)}$ is the field strength renormalization of the field of species χ.

We also introduce β-functions for the dimensionless mass parameters, $\overline{m}^i \equiv m^i/\Lambda$:

$$\beta^j \equiv \Lambda\partial_\Lambda \overline{m}^j = \sum_{i=1}^{\infty} g^{2i}\beta_i^j. \tag{4.22}$$

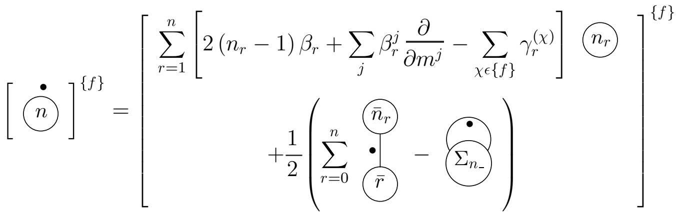


Figure 4. The weak coupling flow equations.

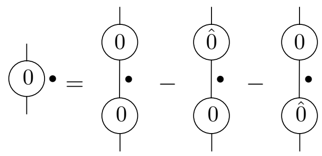


Figure 5. Flow of all possible two-point, classical vertices.

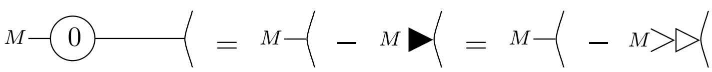
Defining $\Sigma_i = S_i - 2\hat{S}_i$, the weak coupling flow equations follow from substituting (4.18)–(4.22) into the flow equation, as shown in figure 4.

The symbol $\bullet$ means $-\Lambda\partial_\Lambda|_{\overline{m}_u,\overline{m}_c,\ldots}$. We will see shortly why the notation for the ERG kernels, —•—, includes this symbol. A vertex whose argument is an unadorned letter, say n, represents S_n. We define $n_r \equiv n - r$ and $n_\pm \equiv n \pm 1$. The bar notation of the dumbbell term is defined as follows:

$$a_0[\bar{S}_{n-r}, \bar{S}_r] \equiv a_0[S_{n-r}, S_r] - a_0[S_{n-r}, \hat{S}_r] - a_0[\hat{S}_{n-r}, S_r].$$

We illustrate the use of the flow equation by considering the flow of all two-point, classical vertices. This is done by setting $n = 0$ in figure 4 and specializing $\{f\}$ to contain two fields, as shown in figure 5. We note that we can and do choose all such vertices to be single supertrace terms [18, 19].

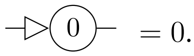
Following [14–19, 32], we use the freedom inherent in $\hat{S}$ by choosing the two-point, classical seed action vertices equal to the corresponding Wilsonian effective action vertices. Figure 5 now simplifies. Rearranging, integrating with respect to Λ and choosing the appropriate integration constants [18, 19], we arrive at the following relationship between the integrated ERG kernels and the two-point, classical vertices.

 (4.23)

We have attached the integrated ERG kernel, denoted by a solid line, to an arbitrary structure since it only ever appears as an internal line. The field labelled by M can be any of the broken phase fields. The object $\blacktriangleright \equiv >\!\triangleright$ is a gauge remainder (cf (1.4)). The gauge remainder components are non-null only in the sectors corresponding to (components of) A^i_μ and F_R and, in these sectors, $\triangleright$ and $>$ are related as a consequence of gauge invariance, as we will see shortly. Note that, in the case that a full gauge remainder bites a vertex, as opposed to just a $\triangleright$, we can replace the half arrows on the right-hand side of (2.18) (which we recall just indicate to former presence of a $\triangleright$) with a $>$ [18, 19].

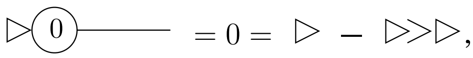
We have been able to construct the effective propagator for each and every independent classical, two-point vertex because we ensured that, for each such vertex, there exists an independent (integrated) kernel.

From the effective propagator relation and (2.20) follows a series of diagrammatic identities. In QCD, as opposed to pure Yang–Mills, the renormalization of $\langle C\rangle_{u,d}$—equivalently the renormalization of the quark masses—means that one-point $C^{1,2}_{u,d}$ vertices exist beyond tree level, spoiling (2.20) at the loop level. The first of the diagrammatic identities is, then, the classical part of (2.20)

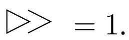


(4.24)

From the effective propagator relation and (4.24), two further diagrammatic identities follow. First, consider attaching an effective propagator to the right-hand field in (4.24) and applying the effective propagator before $\triangleright$ has acted. Diagrammatically, this gives

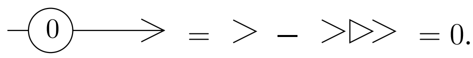


which implies the following diagrammatic identity:

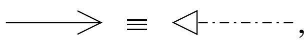


(4.25)

The effective propagator relation, together with (4.25), implies that



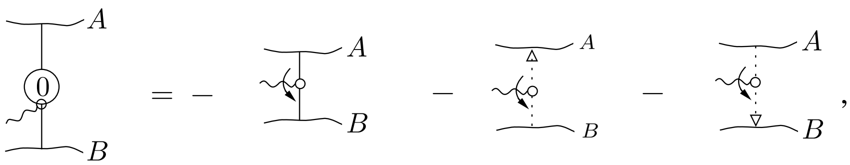
In other words, the (non-zero) structure $\longrightarrow$ kills a classical, two-point vertex. But, by (4.24), this suggests that the structure $\longrightarrow$ must be equal, up to some factor, to $\triangleleft$. Hence,



(4.26)

where the dot-dashed line represents the pseudo-effective propagators of [18, 19].

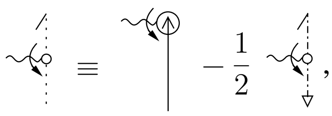
In practice, pseudo-effective propagators only ever appear in a very specific way [25], which we now describe. Consider a three-point vertex attached to two arbitrary structures, A and B, by two effective propagators. The third field on the vertex is taken to be an A^1 carrying momentum p and we suppose that we now Taylor expand the vertex to zeroth order in p. Using (2.21) we have



(4.27)

where the arrow on the momentum derivative symbol indicates in which sense the momentum derivative acts. This is unnecessary in the parent diagram on the left-hand side of (4.27), since it is obvious that the momentum derivative has 'pushed forward' and so corresponds to a derivative with respect to the momentum flowing *into* the vertex, from the structure B. However, once the vertex has been removed via the effective propagator relation, this information is lost, unless explicitly indicated.

Allowing the active gauge remainders in (4.27) to act (according to (2.18)) leads us to define



(4.28)

$$4\beta_1 \Box_{\mu\nu}(p) + \mathrm{O}(p^4) = -\frac{1}{2}$$

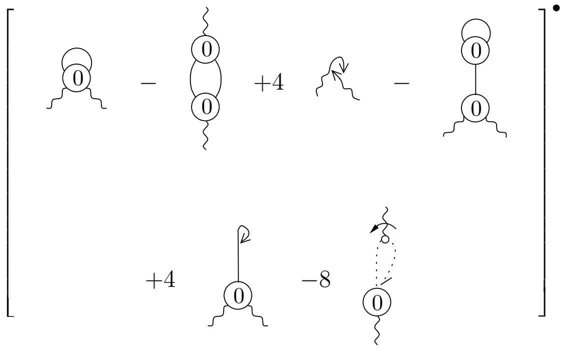


Figure 6. Diagrammatic expression for β_1.

which we will require shortly. Note that the second term on the right-hand side contains a pseudo-effective propagator (differentiated with respect to the momentum flowing into its bottom end).

5. The one-loop β-function

As an illustration of the formalism, we would like to reproduce a standard result, namely the one-loop β-function for massless QCD (we take the massless case since, in our mass-dependent scheme, massive quarks will spoil universality, rendering any comparison with other methods of limited use). Fortunately, due to the developments in [24, 25], this calculation is extremely easy. In [25], a diagrammatic expression has been derived for the n-loop β-function in $SU(N)$ Yang–Mills from which the universal answer (at least at one and two loops) can be directly extracted. The key to deriving this expression is the effective propagator relation, which we have ensured holds for QCD. Indeed, at the one-loop level, the pure $SU(N)$ Yang–Mills expression is exactly the same as in QCD, modulo the changes to the Feynman rules, and it is given in figure 6. (Beyond one loop, the expression of [25] is only slightly modified.) We define $\Box_{\mu\nu}(p) \equiv p^2\delta_{\mu\nu} - p_\mu p_\nu$ and take wiggly lines to denote physical gauge fields (with Lorentz indices suppressed). Note that in $D = 4$ only the second, third and final diagrams contribute, so the expression for β_1 is really very simple.

From comparison with the Yang–Mills (YM) expression [19, 22] and the QED expression [33], it is immediately clear that, as in conventional approaches (though with the number of flavours set equal to twice the number of colours),

$$\beta_1^{\mathrm{QCD}} = \beta_1^{\mathrm{YM}} + 2N\frac{\beta_1^{\mathrm{QED}}}{2}.$$

Setting the quark masses to zero yields

$$\beta_1^{\mathrm{QCD}} = -\frac{N}{(4\pi)^2}\left(\frac{11}{3} - \frac{4}{3}\right).$$

6. Conclusion

We have constructed a manifestly gauge-invariant ERG for QCD and have used it to compute the one-loop β-function for $SU(N)$ Yang–Mills coupled to $2N$ quarks. In the massless limit, we recovered the universal result.

The formalism is a direct extension of the one developed for $SU(N)$ Yang–Mills in [19]. The incorporation of the quarks comprised three steps. First, the quarks had to be added in a way that respected the $SU(N|N)$ regularization scheme. The symmetry associated with the centre of this algebra in fact prevented simply embedding the quarks into the fundamental representation of $SU(N|N)$. Instead, we first embedded sets of N quarks into $N \times N$ matrices, whose rows (columns) were labelled by colour (flavour). Each of these matrices was then embedded into a separate supermatrix valued in complexified $U(N|N)$. In this way, we were able to include multiples of N quarks, with each set of N having degenerate masses.

The second step was to give each of the quarks independent masses, and this required that we broke an unphysical, gauged $SU(N)$ flavour symmetry which is carried by one of the fields belonging to the $SU(N|N)$ regularizing structure. To do this, we introduced Higgs fields for each of the sets of N quarks, whose vevs are essentially mass matrices. The introduction of non-degenerate masses lifts the restriction that the number of flavours must be a multiple of the number of colours, since we are at liberty to remove quarks from the spectrum by tuning their masses to infinity. Clearly, though, the construction is most efficient when, suggestively, the number of flavours is a multiple of the number of colours.

The third and final step was to adapt the ERG equation. This involved not just including additional terms for the additional fields but also defining the covariantization of the ERG kernels appropriately. The key point is that we require the number of independent kernels to be equal to the number of independent, propagating fields. Having given algebraic examples of the covariantizations needed, we wrote the full QCD flow equation in its diagrammatic form. Besides explicitly including the anomalous dimensions of the renormalizing fields, this expression has exactly the same form as that used for pure $SU(N)$ Yang–Mills. Indeed, this similarity allowed us to directly write a diagrammatic expression for the one-loop β-function in QCD. By setting the quark masses to zero we were able to recover the standard, universal result.

The development of a manifestly gauge-invariant ERG for QCD is timely. In $SU(N)$ Yang–Mills, methods exist for computing the expectation values of gauge-invariant operators without fixing the gauge [26] and this work can now be directly generalized to include quarks. We find this particularly exciting in view of the fact that important progress is being made in understanding the structure of non-perturbative contributions both to these expectation values and the β-function [27].

Acknowledgment

OJR acknowledges financial support from PPARC.

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